New Variable Separation Solutions of the (2+1)-Dimensional Generalized Broer-Kaup System

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Using symbolic and algebra computation, the extended tanh-function method (ETM) based on the mapping method is further extended. New variable separation solutions of the (2+1)-dimensional generalized Broer-Kaup (GBK) system are derived. From the periodic wave solution and by selecting appropriate functions, the evolutional behaviours of dromions in the background of Jacobian elliptic wave and their interaction behaviours are investigated.

Key words: Generalized Broer-Kaup (GBK) System; Combined Wave Solutions.
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1. Introduction

Solitons are a manifestation of a balance between inertial and dispersive forces with soliton-supporting equations typically obtained via weakly nonlinear perturbation schemes. However, in higher spatial dimensions these procedures yield equations which typically are not only nonintegrable but, more importantly, rarely support localized patterns. This, strange at first, phenomenon has a simple explanation: whereas nonlinearity due to inertia plays the same role irrespectively of spatial dimension, the increase in degrees of freedom with dimension enhances the dispersive spread and thus tilts the balance. Thus a well balanced model in one dimensional may be less so in higher dimensions. For a genuinely localized structure to emerge in N-D naturally rather than as an exception, the dispersive-inertial balance has to be kept irrespectively of spatial dimension. This may be accomplished either by properly enhancing the inertia, or, as we shall do here, by a proper weakening of the dispersion.

In linear physics, it is generally recognized that the variable separation approach (VSA) is one of the most universal and powerful means for the study of linear partial differential equations (PDEs). The extension of the variable separation approach to nonlinear field has been a highlight. The so-called multilinear variable separation approach (MLVSA) has also been established for various (2+1)-dimensional models [1]. Recently, along with the linear variable separation idea and using the extended tanh-function method (ETM) based on the mapping method, Zheng and some other authors realized the possibility to use the variable separation for some (2+1)-dimensional systems, such as the Broer-Kaup-Kupershmidt (BKK) system [2], the Kortweg-de Vries (KdV) equation [3] and the asymmetric Nizhnik-Novikov-Veselov (NNV) system [4]. Moreover, some authors [5, 6] successfully generalized the ETM to the (1+1)-dimensional and (3+1)-dimensional nonlinear physical models. However, in fact, various solutions including solitary wave solutions, periodic wave solutions and rational function solutions derived by the ETM in [2 – 6], which seem independent, depend on each other. This viewpoint has been proven in [7]. Then, by the nonstandard truncated expansion, Fang and his groups also generalized the ETM and obtained various so-called new symmetrical variable separation solutions for the (2+1)-dimensional BKK [8], Boiti-Leon-Pempinelli (BLP) [9] and NNV [10] systems. However, similarly to our report in [7], we have also proven in [11] that these so-called new solutions in [8 – 10] also depend on each other and the effective solution is identical to the universal formula in [1], which has been given in [7].

More recently, a mapping method [12] is used to realize the variable separation of the (2+1)-dimensional dispersive long wave equation (DLWE). We have also successfully generalized the projective Riccati equation method (PREM) to derive variable separation solutions for some (2+1)-dimensional systems [13]. Moreover, with the help of q-deformed hyperbolic functions, we have also successfully obtained a new variable separation solution for the (2+1)-dimensional KdV equation [14]. Now a natural and important issue is whether we can derive new variable separation solutions via other direct methods. Motivated by this question, we further extend the mapping method and obtain new variable separation solutions of the generalized Broer-Kaup (GBK) system

$$\begin{aligned} &h_t - h_{xx} + 2hh_x + u_x + Au + Bg = 0,\\ &g_t + 2(gh)_x + g_{xx} + 4A(g_x - h_{xy})\\ &+ 4B(g_y - h_{yy}) + C(g - 2h_y) = 0,\\ &u_y - g_x = 0, \end{aligned} \tag{1}$$

where A, B, C are arbitrary constants. The GBK system has recently been derived from a typical (1+1)-dimensional Broer-Kaup (BK) system [15], by means of the Painlevé analysis [16]. When $A = B = C = 0$, the GBK system will degenerate to the celebrated (2+1)-dimensional BK system [17], which can be derived from an inner parameter-dependent symmetry constraint of the Kadomtsev-Petviashvili model [18]. Recently, we also investigated new types of the V-shaped soliton fusion and Y-shaped soliton fission of this system [19].

Shallow water waves and a host of long wave phenomena are commonly investigated by various models of nonlinear evolution equations. Examples include the Korteweg – de Vries, the Camassa-Holm, and the Whitham-Broer-Kaup (WBK) equations. The problem of computing finite amplitude waves on the free surface of an otherwise irrotational fluid has attracted tremendous attention over the years. For the regime of weak nonlinearity and weak dispersion, the Korteweg – de Vries and Boussinesq models have been developed. This paper focuses on a particular form of the higher-order Boussinesq equation, known as the generalized Broer-Kaup equation. The GBK equation is actually an extension of the WBK system using Painlevé analysis [16]. The WBK system is a valuable model for long waves by incorporating or mimicking convective, dispersive and viscous effects. To our knowledge, the evolutional behavior of solitary wave structures in the background of Jacobian elliptic waves is still an open question. In this paper, we will present the evolutional behavior of dromions on the background of a Jacobian elliptic wave. These discussions may help us to comprehend the dynamic problem of spoondrifts in the water waves.

The paper is organized as follows. In Section 2, the ETM is reviewed, and the l-deformed functions are introduced. The variable separation solutions of the (2+1)-dimensional generalized Nizhnik-Novikov-Veselov (GNNV) system are obtained in Section 3. In Section 4, dromions on the background of a Jacobian elliptic wave are discussed, and their interaction behavior is investigated. A brief discussion and summary are given in the last section.

2. Further Extended tanh-Function Method

In the following we would like to outline the main steps of our method:

Given a nonlinear partial differential equation (NPDE), with independent variables $x = (x_0 = t, x_1, x_2, \dots, x_m)$ and a dependent variable u,

$$L(u, u_t, u_{x_i}, u_{x_i x_j}, \cdots) = 0, \tag{2}$$

where L is in general a polynomial function of its arguments, and the subscripts denote the partial derivatives. One assumes that (2) possesses solutions obtained from the ansatz

$$\begin{aligned} u = a_0(x) + \sum_{j=1}^{n} \Big\{ & a_j(x)\phi[R(x)]^j \\ & + b_j(x)\phi[R(x)]^{j-1}\sqrt{l_1 + l_2\phi[R(x)]^2} \Big\}, \end{aligned} \tag{3}$$

with the Riccati equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}R} = l_1 + l_2\phi^2, \tag{4}$$

where $a_0(x)$, $a_j(x)$, $b_j(x)$ and $R(x)$ are arbitrary functions of $x = (x_0 = t, x_1, x_2, \dots, x_m)$ to be determined, l_1 and l_2 are two real constants, and n is fixed by balancing the linear term of the highest-order derivative with the highest-order nonlinear term in (2). To determine u explicitly, one may substitute (3) and (4) into the given NPDE, collect the coefficients of the polynomials in ϕ and $\sqrt{l_1 + l_2\phi[R(x)]^2}$, then eliminate each coefficient to derive a set of partial differential equations for $a_0(x)$, $a_j(x)$, $b_j(x)$ and $R(x)$, solve this system of partial differential equations to obtain $a_0(x)$, $a_j(x)$, $b_j(x)$ and $R(x)$. Finally, (4) possesses the general solutions (without loss of generality, here we only consider the case of $l_1 > 0$):

(i) when $l_1 l_2 = -1$,

$$\phi_1 = l_1 \tanh_{l_1}(R), \tag{5}$$

$$\phi_2 = l_1 \coth_{l_1}(R); \tag{6}$$

(ii) when $l_1 l_2 = 1$,

$$\phi_3 = l_1 \tan_{l_1}(R), \tag{7}$$

$$\phi_4 = l_1 \cot_{l_1}(R); \tag{8}$$

(iii) when $l_1 = 0$,

$$\phi_5 = -\frac{1}{l_2 R}. \tag{9}$$

Moreover, (4) has combined solutions:
(iv) when $l_1 = -l_2 = \frac{1}{2}$,

$$\phi_6 = \tanh(R) \pm \mathrm{i}\,\mathrm{sech}(R), \tag{10}$$

$$\phi_7 = \coth(R) \pm \mathrm{csch}(R); \tag{11}$$

(v) when $l_1 = l_2 = \pm\frac{1}{2}$,

$$\phi_8 = \tan(R) \pm \sec(R), \tag{12}$$

$$\phi_9 = \cot(R) \pm \csc(R). \tag{13}$$

One substitutes $a_0(x)$, $a_j(x)$, $b_j(x)$, $R(x)$ and (5)–(13) into (3), and obtains exact solutions of the given NPDE in concern.

The functions in (5)–(8) are l-deformed functions [20], whose properties will be recalled:

$$\sinh_l(R) = \frac{\mathrm{e}^R - l\mathrm{e}^{-R}}{2}, \quad \cosh_l(R) = \frac{\mathrm{e}^R + l\mathrm{e}^{-R}}{2},$$
$$\tanh_l(R) = \frac{\sinh_l(R)}{\cosh_l(R)}, \quad \mathrm{sech}_l(R) = \frac{1}{\cosh_l(R)}, \tag{14}$$
$$R \in \mathbf{C}.$$

It is straightforward to see that the following formulas hold:

$$(\sinh_l(R))' = \cosh_l(R), \quad (\cosh_l(R))' = \sinh_l(R),$$
$$\cosh_l^2(R) - \sinh_l^2(R) = l, \; (\tanh_l(R))' = l\mathrm{sech}_l^2(R),$$
$$(\mathrm{sech}_l(R))' = -\tanh_l(R)\mathrm{sech}_l(R),$$
$$\tanh_l^2(R) = 1 - l\mathrm{sech}_l^2(R). \tag{15}$$

Correspondingly, we can define l-deformed trigonometric functions as follows:

$$\sin_l(R) = \frac{\mathrm{e}^{\mathrm{i}R} - l\mathrm{e}^{-\mathrm{i}R}}{2\mathrm{i}}, \quad \cos_l(R) = \frac{\mathrm{e}^{\mathrm{i}R} + l\mathrm{e}^{-\mathrm{i}R}}{2},$$
$$\tan_l(R) = \frac{\sin_l(R)}{\cos_l(R)}, \quad \sec_l(R) = \frac{1}{\cos_l(R)}. \tag{16}$$

They satisfy the following formulas

$$(\sin_l(R))' = \cos_l(R), \quad (\cos_l(R))' = -\sin_l(R),$$
$$(\tan_l(R))' = l\sec_l^2(R), \; \sec_l(R))' = \tan_l(R)\sec_l(R),$$
$$\cos_l^2(R) + \sin_l^2(R) = l, \quad 1 + \tan_l^2(R) = l\sec_l^2(R). \tag{17}$$

Remark 1: In [2–7], the authors assume that the ansatz of (2) has the form

$$u = a_0(x) + \sum_{j=1}^{n} a_j(x)\phi(R(x))^j,$$

which merely is a special case of our ansatz (3), when $b_j(x) = 0$. Therefore, we can obtain more new exact solutions of NPDEs by the ansatz (3) in the present paper.

Remark 2: The similar ansatz as (3) has been introduced by Li et al. in [21], however, they merely obtained the travelling solutions by this ansatz. Obviously, our ansatz (3) has a more universal form due to the arbitrary function R of $x = (x_0 = t, x_1, x_2, \ldots, x_m)$. Moreover, the Ricatti equation in this paper has a more universal form than the one in [21].

3. Variable Separation Solutions for the (2+1)-Dimensional GBK System

Now we apply the improved mapping method from section 2 to the (2+1)-dimensional GBK system (1). First, differentiate the first one of (1) with respect to variable y once and substitute the third one into the first one of (1). Then we consider the transformation

$$g = 2h_y, \quad u = 2h_x + \alpha(x,t), \tag{18}$$

where $\alpha(x,t)$ is an arbitrary integrable function of $\{x,t\}$. Equation (1) is reduced to another but equivalent NPDE:

$$h_{ty} + h_{xxy} + 2Ah_{xy} + 2Bh_{yy} + 2(hh_x)_y = 0. \tag{19}$$

By the balancing procedure applied to (19), ansatz (3) becomes

$$h = a_0 + a_1\phi(R) + b_1\sqrt{l_1 + l_2\phi(R)^2}, \tag{20}$$

where $R \equiv R(x,y,t)$, $a_i \equiv a_i(x,y,t)$ $(i=0,1)$, $b_1 \equiv b_1(x,y,t)$, and ϕ satisfies (4). Inserting (20) with (4) into (19), choosing the variable separation ansatz

$$R = p(x,t) + q(y-2Bt), \tag{21}$$

and eliminating all coefficients of the polynomials of ϕ and $\sqrt{l_1 + l_2\phi(R)^2}$, one gets a set of PDEs for $\{a_i \equiv a_i\,(i=0,1), b_1, p, q\}$, from which one can obtain

$$a_0 = -\frac{p_{xx}+p_t+2Ap_x}{2p_x},\quad a_1 = -\frac{l_2}{2}p_x, \quad b_1 = -\frac{\sqrt{l_2}}{2}p_x, \tag{22}$$

where p is an arbitrary function of $\{x,t\}$.

From these results, the variable separation solutions of (19) possess the following form:

Family 1. For $l_1 l_2 = -1$,

$$h_1 = -\frac{p_{xx}+p_t+2Ap_x}{2p_x} + \frac{p_x}{2}\tanh_{l_1}(p+q) + \frac{l_1}{2}\sqrt{l_2}p_x \mathrm{sech}_{l_1}(p+q), \tag{23}$$

$$h_2 = -\frac{p_{xx}+p_t+2Ap_x}{2p_x} + \frac{p_x}{2}\coth_{l_1}(p+q) + \mathrm{i}\frac{l_1}{2}\sqrt{l_2}p_x \mathrm{csch}_{l_1}(p+q). \tag{24}$$

Family 2. For $l_1 l_2 = 1$,

$$h_3 = -\frac{p_{xx}+p_t+2Ap_x}{2p_x} - \frac{p_x}{2}\tan_{l_1}(p+q) + \frac{l_1}{2}\sqrt{l_2}p_x \sec_{l_1}(p+q), \tag{25}$$

$$h_4 = -\frac{p_{xx}+p_t+2Ap_x}{2p_x} - \frac{p_x}{2}\cot_{l_1}(p+q) + \frac{l_1}{2}\sqrt{l_2}p_x \csc_{l_1}(p+q). \tag{26}$$

Family 3. For $l_1 = 0$,

$$h_5 = -\frac{p_{xx}+p_t+2Ap_x}{2p_x} + \frac{p_x}{p+q}. \tag{27}$$

In addition, if we let $l_1 = -l_2 = \frac{1}{2}$ or $l_1 = l_2 = \pm\frac{1}{2}$, we will obtain the combined solutions of (19). Here, we omit these cases for convenience. From (5) – (13), we can obtain rich solutions of the GBK system by the selection of different parameters l_1 and l_2.

From (23) – (27) and (18), we can obtain variable separation solutions of the (2+1)-dimensional GBK system. Especially we are interested in the structure of the periodic wave solution for the field g which has the final form

$$g_3 = 2h_{3y} = -l_1 p_x q_y[\sec_{l_1}(p+q)^2 - \sqrt{l_2}\sec_{l_1}(p+q)\tan_{l_1}(p+q)], \tag{28}$$

where p and q are arbitrary functions of $\{x,t\}$ and $\{y-2Bt\}$, respectively.

Remark 3: It is necessary to point out that all the exact solutions of the (2+1)-dimensional GBK system constructed in this paper have been checked by Maple software. Because of the mapping equation (4), we can't only get new l-deformed hyperbolic function solutions and l-deformed trigonometric function solutions, but also get the combined exact solutions of a class of NPDEs. To our knowledge, these solutions (23) – (26) have not been reported in other literature before. All the solutions [namely (23) – (28)] of the (2+1)-dimensional GBK system obtained in this paper include two independent variables $p(x,t)$ and $q(y-2Bt)$. In these solutions the arbitrary functions imply that (1) has abundant local physical structures.

Remark 4: Although the functions sec and tan have singularities, the arbitrary functions p and q are chosen appropriately to avoid the singularities for g. This property can be found from the discussion of localized structures in the next section. Compared with the solutions in [22], the solutions obtained here have more universal form because of the existence of the combined form. When $b_1 = 0$ in ansatz (20) with $l_2 = 1$, the solutions here degenerate into the solutions in [22].

4. Novel Solitary Wave Structures on the Background of a Jacobian Elliptic Wave

All rich localized coherent structures, such as non-propagating solitons, dromions, peakons, compactons, foldons, instantons, ghostons, ring solitons, and the interactions between these solitons, can be derived by the quantity g expressed by the solitary wave solutions of Family 1 and the variable separation solutions of Family 3. These abundant localized coherent structures are omitted in the present paper since some similar situations have been reported in previous literature [1 – 14]. The case of the periodic wave solution of Family 2 is not discussed in detail, because we expect that the

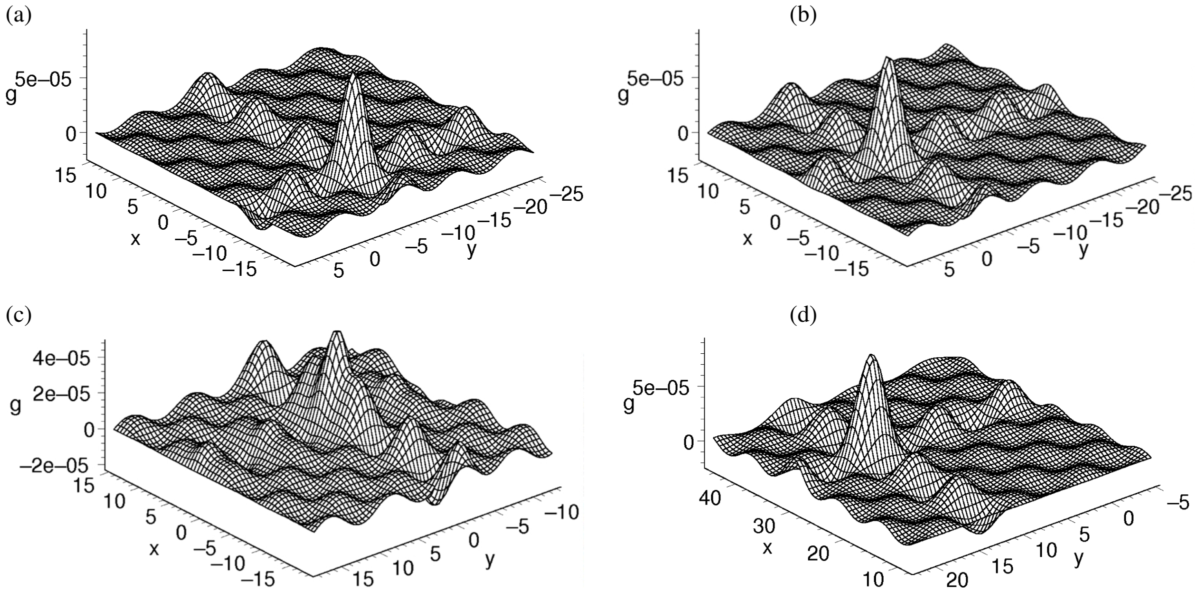


Fig. 1. Evolution graph of a dromion combined on the background of a Jacobian elliptic wave at times: (a) $t = -5$; (b) $t = -1$; (c) $t = 1$; (d) $t = 15$. The corresponding values for the abscissa of the peak position are: (a) $x = -10$; (b) $x = -2$; (c) $x = 2$; (d) $x = 30$.

periodic wave solution will not yield important localized excitations. However, this conjecture may be incorrect. In the following part, without loss of generality, we will discuss some significant localized excitations derived from the periodic wave solution (28) with the special choice $l_1 = l_2 = 1, B = 0.5$, i. e.,

$$g = g_3 = -p_x q_y[\sec(p+q)^2 - \sec(p+q)\tan(p+q)], \quad (29)$$

where p and q are arbitrary functions of $\{x, t\}$ and $\{y - t\}$, respectively.

4.1. Dromions on the Background of a Jacobian Elliptic Wave

Here we focus on this localized structure on the background of a Jacobian elliptic wave, which can be constructed by the choice of p and q as

$$p = -2 - 0.003\mathrm{cn}(0.8x, 0.3) - 0.015\tanh(0.5x - t) \quad (30)$$

and

$$q = 2 + 0.003\mathrm{cn}[-0.8(y - t) + 0.7, 0.3] + 0.015\tanh[0.5(y - t) + 1], \quad (31)$$

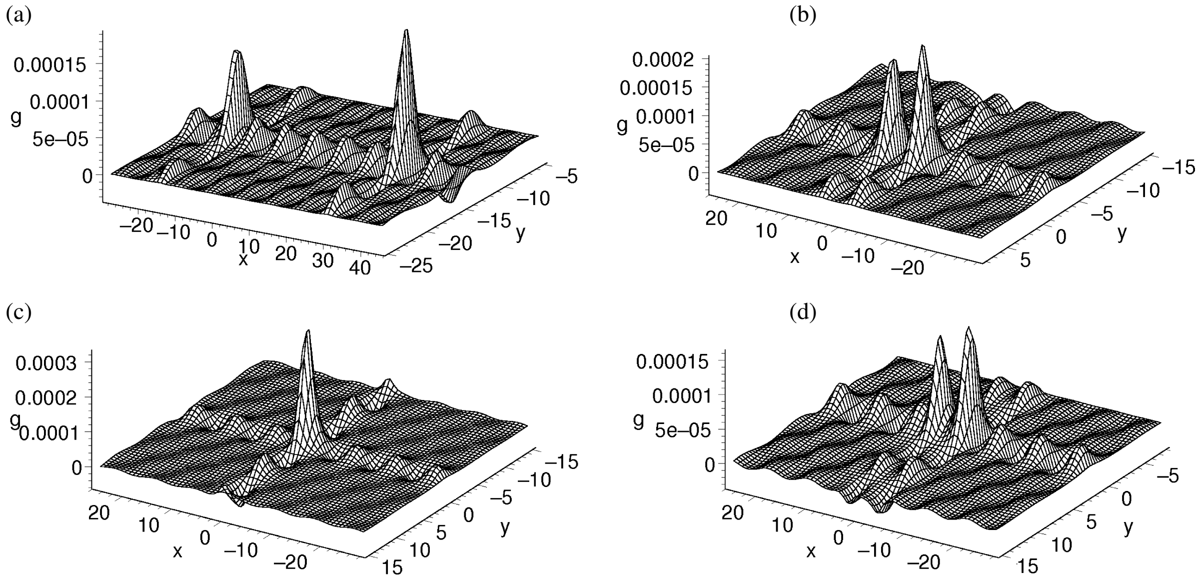
where $\mathrm{cn}(\cdot,\cdot)$ is the Jacobian elliptic cn function with the modulus 0.3.

Figure 1 displays a dromion on the background of a Jacobian elliptic wave which travels along the x-axis. From Fig. 1, we see that the wave amplitude of the dromion changes due to the combination of solitary wave and Jacobian elliptic wave as the background; however, there is no change of velocity. This case is similar to many real physical processes, such as solitons on the water waves.

4.2. Interactions between Dromions on the Background of Jacobian Elliptic Waves

The interaction between solitons can be elastic or inelastic. It is called elastic, if the amplitude, velocity and wave shape of solitons do not change after their interaction. Otherwise, the interaction between solitons is inelastic (incomplete elastic and completely inelastic). Like the collisions between two classical particles, a collision in which the solitons stick together is sometimes called completely inelastic, which is discussed in [8, 19].

We can also investigate the interaction behavior between two dromions on the background of Jacobian el-



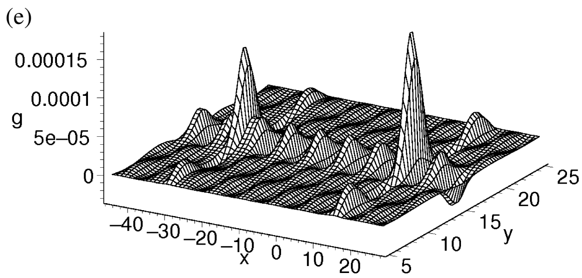


Fig. 2. Interaction evolution graph of two combined dromions on the background of Jacobian elliptic waves at times: (a) $t = -15$; (b) $t = -2$; (c) $t = 0$; (d) $t = 2$; (e) $t = 15$. The corresponding peak abscissa values are: (a) and (e) 0.00012, 0.0002; (b) 0.00019, 0.00025; (c) 0.00035; (d) 0.00014, 0.00016.

liptic waves. When p and q are chosen as

$$p = -2 - 0.003\text{cn}(0.8x, 0.3) \\ -0.015\tanh[0.8(x+2t)] - 0.015\tanh[0.8(x-t)] \tag{32}$$

and

$$q = 2 + 0.003\text{cn}[0.8(y-t), 0.3] \\ + 0.015\tanh[0.8(y-t)], \tag{33}$$

two dromions on the background of Jacobian elliptic waves are obtained, see Figure 2. From Fig. 2, we can see that the interaction between the two dromions is not completely elastic. After interaction, two dromions exchange their shapes completely and preserve their velocities.

More concretely, to see the completely elastic interaction property between two dromions, we cut and move the left ring dromion of Fig. 2e from the center ($[x = -2c_1t_0 + \delta_1,\ y = \delta_2]$ with $t_0 = 15$ and c_1, δ_1 and δ_2 being some suitable constants related to the possible changes of the velocity and the phase shift) to the center of the left dormion of Fig. 2a (before interaction) $[x = -t_0, y = t]$. The resulting single dromion may be described by

$$G_1 \equiv \begin{Bmatrix} g(t = t_0), & x \le 0 \\ 0, & x > 0 \end{Bmatrix}_{\substack{x \to x - (2c_1 - 1)t_0 + \delta_1, \\ y \to y - t + \delta_2}}, \tag{34}$$

where $g(t = t_0)$ is defined by (29) with (30), (31) and $t = t_0 > 0$. Similarly, we cut and move the right dromion of Fig. 2e from $[x = c_2t_0 + \delta_3,\ y = \delta_4]$ to the center of the right dromion of Fig. 2a $[x = 2t_0,\ y = t]$ and the result can be expressed as

$$G_2 \equiv \begin{Bmatrix} 0, & x \le 0 \\ g(t = t_0), & x > 0 \end{Bmatrix}_{\substack{x \to x - (2 - c_2)t_0 + \delta_1, \\ y \to y - t + \delta_2}}. \tag{35}$$

Now choosing the constants c_1, c_2 and $\delta_1 \sim \delta_4$ appropriately to minimize the quantity

$$v \equiv |G_1 + G_2 - g(t = t_0)|, \quad (36)$$

we can find

$$v \approx 0 \quad (37)$$

for

$$c_1 = c_2 = 1 \quad (38)$$

and

$$\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0. \quad (39)$$

The result (37) denotes that the dromions exchange their shapes totally after collision. Equation (38) shows that the dromions preserve their velocities after interaction, and (39) means that there are no phase shifts at all for the head on collision between two dromions.

5. Summary and Discussion

In short, the extended tanh-function method has been improved to obtain variable separation solutions of the (2+1)-dimensional GBK system. Some lower-dimensional arbitrary functions are included in the exact solutions. From the periodic wave solution (28) and by choosing appropriate functions, dromions on the background of Jacobian elliptic waves are discussed, and their interaction behaviours are investigated. We think that the discussions here about solitary waves on the background of Jacobian elliptic waves in higher-dimensional systems are significant and interesting. Of course, there are some pending issues to be further studied. What happens in limiting cases of these new solutions? How to quantify the notion of complete or incomplete elasticity more suitably? What is the measure for the deviation of a solution from elasticity? What is the general equation for the distribution of the energy and momentum for these exotic interactions? How can we use the dromions on the background of Jacobian elliptic waves of integrable models to investigate practically observed solitary wave phenomena in experiments, such as in water waves? Actually, our present short paper is merely the beginning of more extended work. We can obtain even richer exact solutions by a more general ansatz of the NPDE (2), which reads

$$u = a_0(x) + \sum_{j=1}^{n} \Big\{ a_j(x)\phi[R(x)]^j + \frac{b_j(x)}{\phi[R(x)]^j} + c_j(x)\phi[R(x)]^{j-1}\sqrt{l_1 + l_2\phi[R(x)]^2} + \frac{d_j(x)}{\phi[R(x)]^{j-1}\sqrt{l_1 + l_2\phi[R(x)]^2}} \Big\}. \quad (40)$$

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